

Lecture 17: Rotational motion, Kepler's Laws

Some history

Study of the motion of the planets has a special place in history, due to its impact on science and as an example of conflicts that can arise between science and religion. Astronomy has been important since the dawn of civilization and quantitative, scientific studies of the motion of the planets occurred in many early civilizations, for example in the Chinese, Arabian and Incan cultures. Ptolemy (85-160AD) put forward an earth centered (geocentric) model of the motion of the planets. As measurements of planetary motion became more precise, it was clear that Ptolemy's model was wrong. The Polish astronomer Copernicus (1473-1543) put forward a theory based on the idea that planets moved around the sun in circular orbits (heliocentric theory). The Catholic Church had adopted the Ptolemy model as an article of faith as had most other religions and scientists of the time, so Copernicus was considered a heretic. Tycho Brahe made meticulous measurements which supported the heliocentric model of Copernicus which was greatly elaborated upon and supported by Galileo, using his new telescope. Galileo was brought before the Italian inquisition for his belief in the heliocentric model. Kepler showed that circular orbits do not explain the observations and instead proposed his famous three laws, which are now known to be accurate.

Kepler was Lutheran and German and also suffered considerably because of his belief in the heliocentric theory. The study of the motion of the planets provides an important example of the context within which science works, with conflicting theories, interpersonal conflicts, sometimes including conflicts with religion and with the broader society. Another truth is old theories never really die and by surfing the internet, you can still find sites vacuously articulating opposition to the heliocentric model and supporting the view held by the majority of scientists and religious figures of the sixteenth century. Another amusing topic is the flat earth theory, which was actually rejected by most intellectuals of the later Greek and Roman civilizations. It was somewhat popular in the very early middle ages but was not popular at the time of Columbus, though many people incorrectly believe that it was religious dogma at the time. Of course evolution is a much more topical conflict between science and religion in our society.

Kepler's laws

- (i) All planets move in elliptical orbits with the sun as one of the foci.
- (ii) A line drawn to the sun from a planet sweeps out equal areas in equal times.
- (iii) $T^2 \propto r^3$, where T is orbital period and r is the average distance from the planet to the sun.

Kepler's laws apply to a small mass (e.g. a planet) moving around a large mass (e.g. the sun). They apply to moons moving around planets, asteroids moving around the sun etc. Here we will consider the special case of circular orbits and demonstrate that if (i) is true, the (ii) and (iii) follow. In later lectures, we shall return to the case of elliptical orbits. Consider a planet moving in a circular orbit of radius r with constant speed v as seen from the sun. If the orbital speed is constant and the motion is circular, then no proof of (ii) is needed.

Proof of Kepler's third law follows from Newton's second law and the formula for centripetal acceleration,

$$\frac{GMm}{r^2} = mv^2/r \quad (1)$$

We find the relation to the period, T , by using velocity = distance/time, so that,

$$v = \frac{2\pi r}{T} \quad (2)$$

Using this to remove v and cancelling the mass, m , we find that,

$$T^2 = \left(\frac{4\pi^2}{GM}\right)r^3 = Kr^3 \quad (3)$$

For the case of the sun, the constant K becomes $K_s = (4\pi^2/GM_s) = 2.97 * 10^{-19} s^2/m^3$. It is possible to find the mass of the sun by knowing T and r , both of which can be deduced from observations of planetary motion. Furthermore, similar methods can be used to deduce the mass of any planet which has a moon. The period of the moon can be measured and the radius of the orbit can be measured when the moon is adjacent the planet.

A couple of problems

Problem: Held up in a rotating cannister. Given a static friction coefficient of $\mu_s = 0.5$, find the angular velocity required to prevent a person slipping down the side of a cylindrical cannister of radius $r = 5m$, when the cannister is vertical.

Solution: The acceleration required to produce circular motion is v^2/r . The associated force is mv^2/r . The normal force of the person on the cannister is $N = mv^2/r$, so the friction force is $\mu_s N = \mu_s mv^2/r$. The force of gravity on the person is mg , so the rotational speed required to resist the downward force of gravity through friction is given by,

$$\mu_s m \frac{v^2}{r} = \mu_s m \omega^2 r = mg \quad (4)$$

so that,

$$v = \left(\frac{gr}{\mu_s}\right)^{1/2} = 9.9m/s \quad (5)$$

Which implies that $\omega = v/r = 1.95rad/s = 18.6rev/min$. At fixed angular velocity, the larger the radius the greater the force.

Problem: Where the rubber meets the road.

(i) Cornering depends on centripetal acceleration and on friction. There are many variants of these problems. First consider using only friction to corner. What friction force is supplied by the tyres on the road? This is easy enough,

$$f_s = mv^2/r \quad (6)$$

We might also be asked what value of μ_s is required, in which case we can use $f_s = \mu_s mg = mv^2/r$, so that,

$$\mu_s = \frac{v^2}{rg} \quad (7)$$

Note that this is independent of mass, so that heavy cars and light cars need the same quality tyres at least when cornering. Other interesting issues to do with friction are related to braking. For example the coefficient of static friction between a good dry asphalt road and tyres is about $\mu_s = 0.8$, while on a wet road it is a half that or smaller. Also, since the static friction coefficient is higher than the kinetic friction coefficient, it is better to brake without skidding.

(ii) A second variant of these problems is to reduce the friction force required of the tyres by “banking” the turns. For example in NASCAR: Daytona, where the average qualifying speed is about 200mph the tightest turn is banked at about 31 degrees. At michigan International speedway, average speed about 180mph, the tightest turn is banked at about 18 degrees. Highspeed trains also have banked turns and of course roller coasters have the ultimate in banked turns, including upside down.

A simple case is to ask: In a spiral roller coaster with radius $r = 4m$, which is the minimum angular velocity required to prevent the roller coaster from falling off the track? At the top of the circle, there is an acceleration g downward. This acceleration can produce circular motion, or it can cause the coaster to fall. In order for it to produce circular motion, we need to be moving fast enough, ie

$$g = \frac{v^2}{r} \quad (8)$$

The smallest velocity is then, $v_{min} = (9.81 * 4)^{1/2} = 6.27m/s$. At this speed you would feel weightless at the top of the ride. Most spiral coasters move at speeds a factor of two or so faster than this.

Back to the racetrack. If we bank a turn, then there is a component of the normal force which is in the radial direction. If the corner has a banking angle of θ , then the component of the force toward the center of the circle is

$$N \sin(\theta) = \frac{mv^2}{r} \quad (9)$$

To find the normal force, we assume that friction is absent, in that case we have, $N \cos(\theta) = mg$. [Note that this is different than the case where we considered a block on an inclined plane where we took the component of the gravitational force along the direction of the normal force. There is no contradiction here, as can be demonstrated by carefully carrying out the calculation with the friction force included.] In the case we are studying here, we solve for $N = mg / \cos(\theta)$ and plug into Eq. (9) to find,

$$\tan(\theta) = \frac{v^2}{rg} \quad (10)$$

This angle is the special angle at which no tyre friction is required during cornering. It is all supplied by the component of the normal force due to the

banked turn. Note that the special banking angle does not depend on the mass of the car. If a car goes around the corner at a speed which is slower than that given by this formula, it will tend to slip down the banked corner, while if a car goes faster it tends to slide up the turn. Friction by car tyres can handle this additional force provided it is not too large.